

A REMARK ON THE WIENER-IKEHARA TAUBERIAN THEOREM

RYO KATO

Abstract. In this paper we point out that the proof of Kable's extension of the Wiener-Ikehara Tauberian theorem can be applied to the case where the Dirichlet series has a pole of order " l/m " without much modification (Kable proved the case $l = 1$).

1. INTRODUCTION

We use the notation $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ for the sets of positive integers, all integers, real numbers, complex numbers respectively. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. For $x \in \mathbb{R}$ we put $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Let $\{a_n\}$ be a sequence of non-negative real numbers, d a positive real number, and $m, l \in \mathbb{N}$. Suppose that the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\operatorname{Re}(s) > d$. Also suppose that L^m has a meromorphic continuation to an open set containing the closed half-plane $\operatorname{Re}(s) \geq d$ and holomorphic except for a pole of order l at $s = d$. In this paper, if L^m has a pole of order l , then we say that L has a pole of order l/m . For functions f and $g : \mathbb{R} \rightarrow \mathbb{R}$, we denote $f(x) \sim g(x)$ if $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$.

Our purpose is to determine the asymptotic behavior of $\sum_{n \leq X} a_n$ as $X \rightarrow \infty$ by properties of $L(s)$.

In the case where L has a simple pole at $s = d$ (i.e., $m = l = 1$), Wiener and Ikehara proved that

$$\sum_{n \leq X} a_n \sim \frac{A}{d} X^d$$

as $X \rightarrow \infty$, where A is residue of L at $s = d$. This result was proved in 1932([6]) and is called the Wiener-Ikehara theorem.

In the general case, an extension was given by Delange ([1, p.235, THÉORÈM III]) in 1954. Delange considered the case where the order of the pole is a positive real number in some sense. However, to apply his theorem to L satisfying above conditions, an extra condition about zeros of L^m on $\operatorname{Re}(s) = d$ is required. Kable has given an extension for the case where the order of the pole is $1/m$ without the condition. In [2], he used the notion of functions of bounded variation.

The result of Delange and Kable is as follows. Let $\alpha = 1/m$ or l . If $\alpha = 1/m$, then suppose that the residue of L^m at $s = d$ is A^m , where $A > 0$. If $\alpha = l$, then let

2010 *Mathematics Subject Classification.* 11M45.

Key words and phrases. Density theorem, Wiener-Ikehara theorem.

$A = \lim_{s \rightarrow d} L(s)(s - d)^l$ ($A > 0$). Then

$$(1.1) \quad \sum_{n \leq X} a_n \sim \frac{AX^d}{d\Gamma(\alpha)(\log(X))^{1-\alpha}}$$

as $X \rightarrow \infty$. We shall point out in this paper that the proof of Kable's result works for the case, without much modification, where the order of the pole is l/m , and that (1.1) holds with $\alpha = l/m$.

The organization of this paper is as follows. In Section 2, we define symbols and functions used in this paper. Then we slightly extend [2, p.140, THEOREM 1]. In Section 3, we apply the result in Section 2 to obtain the same result as (1.1) for the case $\alpha = l/m$. In Section 4, we give an example of a Dirichlet series which has a pole of order $2/3$ and apply the main theorem (Theorem 3.1).

2. PRELIMINARIES

Let λ be $(2\pi)^{-\frac{1}{2}}$ times the Lebesgue measure on \mathbb{R} and $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on \mathbb{R} . For $\Phi \in \mathcal{S}(\mathbb{R})$ define the Fourier transform by

$$\mathcal{F}(\Phi)(t) = \int_{\mathbb{R}} \Phi(u) e^{-iut} d\lambda(u).$$

We define the inverse Fourier transform by

$$\mathcal{F}^{-1}(\Phi)(t) = \int_{\mathbb{R}} \Phi(u) e^{iut} d\lambda(u).$$

Then

$$\mathcal{F}^{-1}(\mathcal{F}(\Phi))(t) = \Phi(t) \quad \text{i.e.,} \quad \Phi(t) = \int_{\mathbb{R}} \mathcal{F}(\Phi)(u) e^{iut} d\lambda(u).$$

For $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$, we define the convolution by

$$(\Phi * \Psi)(t) = \int_{\mathbb{R}} \Phi(t - u) \Psi(u) d\lambda(u).$$

It is well known (see [5, p.183, 7.2 Theorem]) that

$$\mathcal{F}(\Phi * \Psi) = \mathcal{F}(\Phi) \mathcal{F}(\Psi).$$

For $s \in \mathbb{C} - (-\infty, 0]$ we choose the branch of $\log(s)$ so that $-\pi < \arg(s) < \pi$. Let $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$. For $s \in \mathbb{C} - \{s \in \mathbb{C} \mid s - x \in (\infty, 0]\}$ we define $(s - x)^\alpha = \exp(\alpha \log(s - x))$. Then $(s - x)^\alpha$ is holomorphic on $\mathbb{C} - \{s \in \mathbb{C} \mid s - x \in (\infty, 0]\}$ and has positive real values for $s \in \{s \in \mathbb{C} \mid s - x \in \mathbb{R}_{>0}\}$.

Let $\mathcal{P}([a, b])$ denote the set of all partitions of $[a, b]$ (i.e., sequences $x_0 = a < x_1 < \dots < x_n = b$). For a function $f : [a, b] \rightarrow \mathbb{R}$ we define $V_a^b f \in \mathbb{R} \cup \{\infty\}$ by

$$V_a^b f = \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \mid P = \{x_i \mid i = 0, 1, \dots, n\} \in \mathcal{P}([a, b]) \right\}.$$

We say that a function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation if $V_a^b f < +\infty$. Also we say that a function $f : [a, b] \rightarrow \mathbb{C}$ is of bounded variation if the real part and the imaginary part of f are of bounded variation. If a function $f : \mathbb{R} \rightarrow \mathbb{C}$ is of bounded variation on any closed-interval $[a, b]$, then we say that f is locally of bounded variation.

The following theorem is an extension of the Wiener-Ikehara theorem and plays a key role to prove the main theorem in Section 3. This theorem slightly extends [2, p.140, THEOREM 1].

Theorem 2.1. *Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a non-decreasing function such that the Laplace transform*

$$F(s) = \int_0^\infty f(u)e^{-su} du$$

converges absolutely for $\operatorname{Re}(s) > 1$. Let $\alpha, \alpha_i \in \mathbb{R}_{>0}$ ($i = 1, 2, \dots, r$), $A \in \mathbb{R}_{>0}$, $A_i \in \mathbb{C}$ and $\alpha > \alpha_i$ for all i . We define

$$G(s) = F(s) - \frac{A}{(s-1)^\alpha} - \sum_{i=1}^r \frac{A_i}{(s-1)^{\alpha_i}}$$

for $\operatorname{Re}(s) > 1$. Suppose that the function G extends continuously to the set $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 1\}$. If $\alpha < 1$ then assume, in addition, that once so extended, the function $t \mapsto G(1+it)$ is locally of bounded variation. Then

$$\lim_{u \rightarrow \infty} u^{1-\alpha} e^{-u} f(u) = \frac{A}{\Gamma(\alpha)}.$$

Proof. Define a function $h : \mathbb{R} \rightarrow [0, \infty)$ by $h(u) = u^{1-\alpha} e^{-u} f(u)$ and let $C_i = A_i / \Gamma(\alpha_i)$, $C = A / \Gamma(\alpha)$. By [2, p.140, LEMMA 2]

$$\frac{1}{(s-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(s-1)u} u^{\alpha-1} du$$

for $\operatorname{Re}(s) > 1$. By definition

$$F(s) = \int_0^\infty h(u) e^{-(s-1)u} u^{\alpha-1} du.$$

It follows that

$$G(s) = \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) e^{-(s-1)u} u^{\alpha-1} du$$

for $\operatorname{Re}(s) > 1$. Let Ψ be an even Schwartz function with compact support and $\Phi = \mathcal{F}(\Psi)^2$. Since $\mathcal{F}(\Phi) = \Psi * \Psi$, $\mathcal{F}(\Phi)$ is even and has compact support.

Suppose that $\sigma > 1$ and $x > 0$. Since

$$F(\sigma) \geq \int_x^\infty f(u) e^{-\sigma u} du, \quad f(u) \geq 0$$

and $f(u)$ is monotone increasing, we obtain $f(x) \leq \sigma e^{\sigma x} F(\sigma)$. Hence,

$$(2.2) \quad h(x) \leq \sigma F(\sigma) e^{(\sigma-1)x} x^{1-\alpha}$$

for any $x > 0$.

Let $\varepsilon > 0$ be a small number. We choose σ so that $0 < \sigma - 1 < \varepsilon$. For any $v \in \mathbb{R}_{>0}$, by using (2.2), we obtain

$$(2.3) \quad \int_0^\infty h(u) \Phi(v-u) e^{-\varepsilon u} u^{\alpha-1} du \leq \sigma F(\sigma) \int_0^\infty \Phi(v-u) e^{-(\varepsilon-(\sigma-1))u} du.$$

Since Φ is bounded and non-negative, the integral on the right-hand side of (2.3) converges. Since $h(u)$ is also non-negative, the integral on the left-hand side converges absolutely. Moreover the integral

$$\int_0^\infty \Phi(v-u)e^{-\varepsilon u}u^{\alpha_i-1}du$$

also converges absolutely. Therefore, by Fubini's theorem, the following equation holds.

$$\begin{aligned} & \int_{\mathbb{R}} \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) \Phi(v-u)e^{-\varepsilon u}u^{\alpha-1}du \\ (2.4) \quad &= \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) e^{-\varepsilon u}u^{\alpha-1} \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{i(v-u)t}d\lambda(t)du \\ &= \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt} \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) e^{-(it+\varepsilon)u}u^{\alpha-1}dud\lambda(t) \\ &= \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+\varepsilon+it)d\lambda(t). \end{aligned}$$

By assumption, $G(s)$ is continuous on $\operatorname{Re}(s) \geq 1$ and the support of $\mathcal{F}(\Phi)$ is compact. Thus, $G(1+\varepsilon+it)$ converges to $G(1+it)$ uniformly on the support of $\mathcal{F}(\Phi)(t)$ as $\varepsilon \rightarrow +0$. Therefore,

$$(2.5) \quad \lim_{\varepsilon \rightarrow +0} \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+\varepsilon+it)d\lambda(t) = \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t).$$

As far as the left-hand side of (2.4) is concerned, by using the monotone convergence theorem to each term, we have

$$\begin{aligned} (2.6) \quad & \lim_{\varepsilon \rightarrow +0} \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) \Phi(v-u)e^{-\varepsilon u}u^{\alpha-1}du \\ &= \int_{\mathbb{R}} \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) \Phi(v-u)u^{\alpha-1}du. \end{aligned}$$

We may choose a closed-interval $[a, b]$ containing the support of $\mathcal{F}(\Phi)$. By [2, p.140 LEMMA 3], if $\alpha < 1$

$$\begin{aligned} (2.7) \quad & \lim_{v \rightarrow \infty} v^{1-\alpha} \int_{\mathbb{R}} \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t) \\ &= \lim_{v \rightarrow \infty} v^{1-\alpha} \int_a^b \mathcal{F}(\Phi)(t)e^{ivt}G(1+it)d\lambda(t) \\ &= 0. \end{aligned}$$

If $\alpha \geq 1$, then we have the same equation by [5, p.185, 7.5 Theorem]. Since the right-hand sides of (2.5) and (2.6) are equal, by using (2.7), we obtain

$$(2.8) \quad \lim_{v \rightarrow \infty} v^{1-\alpha} \int_0^\infty \left(h(u) - C - \sum_{i=1}^r \frac{C_i}{u^{\alpha-\alpha_i}} \right) \Phi(v-u)u^{\alpha-1}du = 0$$

for any $\alpha > 0$.

By [2, p.139, Lemma 1] we have

$$(2.9) \quad \lim_{v \rightarrow \infty} v^{1-\alpha} \int_0^\infty \Phi(v-u) u^{\alpha-1} du = \int_{\mathbb{R}} \Phi(u) du$$

for any $\alpha > 0$. So,

$$(2.10) \quad \begin{aligned} & \lim_{v \rightarrow \infty} v^{1-\alpha} \int_0^\infty \frac{C_i}{u^{\alpha-\alpha_i}} \Phi(v-u) u^{\alpha-1} du \\ &= \lim_{v \rightarrow \infty} \frac{1}{v^{\alpha-\alpha_i}} v^{1-\alpha_i} \int_0^\infty C_i \Phi(v-u) u^{\alpha_i-1} du \\ &= 0 \end{aligned}$$

for $i = 1, 2, \dots, r$. Therefore, by (2.8), (2.9), and (2.10), we have

$$\begin{aligned} \lim_{v \rightarrow \infty} v^{1-\alpha} \int_0^\infty h(u) \Phi(v-u) u^{\alpha-1} du &= \lim_{v \rightarrow \infty} C v^{1-\alpha} \int_0^\infty \Phi(v-u) u^{\alpha-1} du \\ &= C \int_{\mathbb{R}} \Phi(u) du. \end{aligned}$$

The rest of the argument is similar to that in [2, pp.142-143], and we can conclude that

$$\lim_{u \rightarrow \infty} h(u) = C.$$

Therefore, Theorem 2.1 is proved. \square

3. MAIN THEOREM

In this section, we apply Theorem 2.1 to Dirichlet series which are obtained by sequences of non-negative real numbers and satisfy some conditions. The following theorem is the main application of Theorem 2.1.

Theorem 3.1. *Let $\{a_n\}$ be a sequence of non-negative real numbers, $d \in \mathbb{R}_{>0}$ and m a positive integer. Suppose that the Dirichlet series*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\operatorname{Re}(s) > d$. Also suppose that L^m has a meromorphic continuation to an open set containing the closed half-plane $\operatorname{Re}(s) \geq d$ and holomorphic except for a pole of order l at $s = d$ with $\lim_{s \rightarrow d} L(s)^m (s-d)^l = A^m$, where $A > 0$. Then we have

$$\sum_{n \leq X} a_n \sim \frac{AX^d}{d\Gamma(l/m)(\log(X))^{1-\frac{l}{m}}}$$

as $X \rightarrow \infty$.

Proof. For a subset $S \subset \mathbb{R}$, let ϕ_S be the characteristic function of S . Define a function $f : \mathbb{R} \rightarrow [0, \infty)$ by

$$f(u) = \sum_{n=1}^{\infty} a_n \phi_{[d \log(n), \infty)}(u).$$

By direct computation, we have

$$F(s) = \int_0^\infty f(u) e^{-su} du = \frac{1}{s} L(ds)$$

for $\operatorname{Re}(s) > 1$.

Let $L(s)^m(s-d)^l = Q(s)$. Since $L(s)^m$ has a pole of order l at $s = d$, $Q(s)$ is holomorphic around $s = d$ and $Q(d) = A^m (\neq 0)$. So there exists a holomorphic function $P(s)$ defined on an open disc D with center at $s = d$ such that $P(s)^m = Q(s)$. Since $L(s)(s-d)^{\frac{l}{m}}$ is holomorphic on $\operatorname{Re}(s) > d$, there exists an m -th root of unity ζ such that $L(s)(s-d)^{\frac{l}{m}} = \zeta \cdot P(s)$ on $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > d\} \cap D$. Therefore, $L(s)(s-d)^{\frac{l}{m}}$ can be extended to a holomorphic function around $s = d$, and $\lim_{s \rightarrow d} L(s)(s-d)^{\frac{l}{m}} = A$ ($A \in \mathbb{R}_{>0}$) since $L(s)(s-d)^{\frac{l}{m}}$ has positive real values for $s \in \mathbb{R}_{>d}$. Hence, $\frac{1}{s}L(ds)(ds-d)^{\frac{l}{m}}$ is holomorphic around $s = 1$. Since

$$F(s)(s-1)^{\frac{l}{m}} = \frac{1}{s}L(ds)(s-1)^{\frac{l}{m}} = \frac{1}{sd^{\frac{l}{m}}}L(ds)(ds-d)^{\frac{l}{m}},$$

$F(s)(s-1)^{\frac{l}{m}}$ is holomorphic around $s = 1$.

Therefore, there exists a holomorphic function $B(s)$ defined on an open disc D' with center at $s = 1$ such that

$$F(s)(s-1)^{\frac{l}{m}} = \sum_{i=0}^r A_i(s-1)^i + (s-1)^{r+1}B(s),$$

where $r = \lceil l/m \rceil - 1$ and $A_i \in \mathbb{C}$ for $i = 0, 1, \dots, r$. Then

$$\lim_{s \rightarrow 1} F(s)(s-1)^{\frac{l}{m}} = \frac{A}{d^{\frac{l}{m}}} \quad \text{i.e.,} \quad A_0 = \frac{A}{d^{\frac{l}{m}}}.$$

Thus, we have

$$F(s) = \frac{A_0}{(s-1)^{\frac{l}{m}}} + \sum_{i=1}^r \frac{A_i}{(s-1)^{\frac{l}{m}-i}} + (s-1)^{r+1-\frac{l}{m}}B(s)$$

for $s \in D' \cap \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$.

Now, we define

$$\begin{aligned} G(s) &= F(s) - \frac{A_0}{(s-1)^{\frac{l}{m}}} - \sum_{i=1}^r \frac{A_i}{(s-1)^{\frac{l}{m}-i}} \\ &= \frac{1}{s}L(ds) - \frac{A_0}{(s-1)^{\frac{l}{m}}} - \sum_{i=1}^r \frac{A_i}{(s-1)^{\frac{l}{m}-i}} \end{aligned}$$

for $\operatorname{Re}(s) > 1$.

We shall prove the following (a), (b).

- (a) The function $G(s)$ extends continuously to the set $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 1\}$.
- (b) The function $x \mapsto G(1+ix)$ ($x \in \mathbb{R}$) is locally of bounded variation.

Then $f(s)$, $F(s)$, and $G(s)$ satisfy the condition of Theorem 2.1. It is enough to prove (a), (b) locally.

We first consider the neighborhood of $s = 1$. By definition

$$G(s) = (s-1)^{r+1-\frac{l}{m}}B(s)$$

for $s \in D' \cap \{s \in \mathbb{C} \mid \operatorname{Re}(s) > d\}$.

By Theorems 10.2 of [3, p.192], the function $x \mapsto B(1 + ix)$ is of bounded variation on any closed-interval $[a, b]$, where $\{1 + ix \in \mathbb{C} \mid x \in [a, b]\} \subset D'$. For any $\alpha > 0$ we can obviously extend $(s - 1)^\alpha$ to a continuous function on $\text{Re}(s) \geq 1$. By a similar argument as in [2, p.144], the function $x \mapsto ((1 + ix) - 1)^\alpha$ ($x \in \mathbb{R}$) is locally of bounded variation.

Therefore, if $r + 1 - \frac{1}{m} > 0$, then $G(s)$ extends continuously to the set $D' \cap \{s \in \mathbb{C} \mid \text{Re}(s) \geq 1\}$ and the function $x \mapsto G(1 + ix)$ ($x \in \mathbb{R}$) is of bounded variation on any closed-interval $[a, b]$, where $\{1 + ix \in \mathbb{C} \mid x \in [a, b]\} \subset D'$. If $r + 1 - \frac{1}{m} = 0$, then $G(s)$ has the same properties since $G(s) = B(s)$ on $D' \cap \{s \in \mathbb{C} \mid \text{Re}(s) > 1\}$. Thus, (a), (b) are proved around $s = 1$.

Next we consider the neighborhood of $s = 1 + ix_0$, where $x_0 \in \mathbb{R} - \{0\}$. By the definition of $G(s)$, it is enough to prove (a), (b) for $L(ds)$ instead of $G(s)$.

Since L^m is holomorphic around $s = d + ix_0$, there exists a holomorphic function $R(s)$ defined for s near $d + ix_0$ such that

$$L(s)^m = (s - (d + ix_0))^k R(s), \quad R(d + ix_0) \neq 0, \quad \text{and} \quad k \geq 0.$$

Since $R(d + ix_0) \neq 0$, there exists a holomorphic function $T(s)$ defined on an open disc D'' with center at $s = d + ix_0$ such that $T(s)^m = R(s)$. Then, there exists an m -th root of unity ζ such that

$$L(s) = \zeta \cdot (s - (d + ix_0))^{\frac{k}{m}} T(s)$$

on $D'' \cap \{s \in \mathbb{C} \mid \text{Re}(s) > d\}$.

Now, for any $x_0 \in \mathbb{R} - \{0\}$ we can extend $(s - (d + ix_0))^{\frac{k}{m}}$ to a continuous function on $\text{Re}(s) \geq d$ and the function $x \mapsto (d + ix - (d + ix_0))^{\frac{k}{m}}$ is locally of bounded variation also as in [2, p.144]. Thus, $L(s)$ extends continuously to the set $D'' \cap \{s \in \mathbb{C} \mid \text{Re}(s) \geq d\}$ and the function $x \mapsto L(d + ix)$ is of bounded variation around x_0 . It follows that we can extend $L(ds)$ to a continuous function on $\text{Re}(s) \geq 1$ except $s = 1$ and the function $x \mapsto L(d + idx)$ is of bounded variation on any closed-interval $[a, b]$ not containing $x = 0$. Thus, we have proved (a), (b).

Therefore, we can apply Theorem 2.1 to $f(s)$, $F(s)$, $G(s)$, and $\alpha = l/m$. So we have

$$(3.11) \quad \lim_{u \rightarrow \infty} u^{1-\frac{l}{m}} e^{-u} f(u) = \frac{A_0}{\Gamma(l/m)} = \frac{A}{d^{\frac{l}{m}} \Gamma(l/m)}.$$

Since $f(d \log X) = \sum_{n \leq X} a_n$, by substituting $d \log X$ for u in (3.11), we have

$$\begin{aligned} \lim_{X \rightarrow \infty} (d \log X)^{1-\frac{l}{m}} e^{-(d \log X)} f(d \log X) &= \frac{A}{d^{\frac{l}{m}} \Gamma(l/m)}, \\ \lim_{X \rightarrow \infty} d^{1-\frac{l}{m}} (\log X)^{1-\frac{l}{m}} X^{-d} \sum_{n \leq X} a_n &= \frac{A}{d^{\frac{l}{m}} \Gamma(l/m)}. \end{aligned}$$

Therefore,

$$\sum_{n \leq X} a_n \sim \frac{AX^d}{d\Gamma(l/m)(\log(X))^{1-\frac{l}{m}}}.$$

□

4. AN EXAMPLE

We give an example of a Dirichlet series having a pole of order "2/3" and apply Theorem 3.1.

If $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is a prime decomposition of a positive integer n where p_1, \dots, p_r are distinct primes, then we define $f(n) = \sum_{i=1}^r e_i$. We define a Dirichlet series L by

$$L(s) = \prod_{p:\text{prime}} \left(1 - \frac{2}{3}p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{f(n)} n^{-s}.$$

Then the function $L(s)$ is holomorphic on $\text{Re}(s) > 1$ since $(2/3)^{f(n)} \leq 1$. We claim that $L(s)^3$ has a meromorphic continuation to an open set containing $\text{Re}(s) \geq 1$ and holomorphic except for a pole of order 2 at $s = 1$.

By computation

$$L(s)^3 = \prod_{p:\text{prime}} \left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)^{-1},$$

and

$$\begin{aligned} \frac{L(s)^3}{\zeta(s)^2} &= \prod_{p:\text{prime}} \frac{(1 - p^{-s})^2}{\left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)} \\ &= \prod_{p:\text{prime}} \frac{(1 - p^{-s})^2(1 + p^{-s})^2}{\left(1 - 2p^{-s} + \frac{4}{3}p^{-2s} - \frac{8}{27}p^{-3s}\right)(1 + p^{-s})^2} \\ &= \prod_{p:\text{prime}} \frac{(1 - p^{-2s})^2}{\left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right)} \\ &= \zeta(2s)^{-2} \prod_{p:\text{prime}} \left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right)^{-1}. \end{aligned}$$

Let

$$F(s) = \prod_{p:\text{prime}} \left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s}\right).$$

If $\text{Re}(s) \geq 0$, then we have

$$\left| -\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s} \right| \leq \frac{83}{27}|p^{-2s}|.$$

Thus, if $\text{Re}(s) > 1/2$, then

$$\begin{aligned} \sum_{p:\text{prime}} \left| -\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s} \right| &\leq \sum_{p:\text{prime}} \frac{83}{27}p^{-2\text{Re}(s)} \\ &< +\infty. \end{aligned}$$

Therefore,

$$\sum_{p:\text{prime}} \left| -\frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s} \right|$$

converges absolutely and uniformly on any compact subset of the open half-plane $\operatorname{Re}(s) > 1/2$. It follows that

$$\prod_{p:\text{prime}} \left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s} \right)$$

converges uniformly on any compact subset of the open half-plane $\operatorname{Re}(s) > 1/2$ and is holomorphic on $\operatorname{Re}(s) > 1/2$ (see [4, p.300, 15.6 Theorem]). Thus, $F(s)$ is holomorphic on $\operatorname{Re}(s) > 1/2$.

For any prime number p , since $(1 - \frac{2}{3}p^{-s}), (1 + p^{-s}) \neq 0$ on $\operatorname{Re}(s) > 1/2$,

$$\left(1 - \frac{5}{3}p^{-2s} + \frac{10}{27}p^{-3s} + \frac{20}{27}p^{-4s} - \frac{8}{27}p^{-5s} \right) = (1 + p^{-s})^2 \left(1 - \frac{2}{3}p^{-s} \right)^3 \neq 0$$

on $\operatorname{Re}(s) > 1/2$. Hence $F(s) \neq 0$ on $\operatorname{Re}(s) > 1/2$ i.e., $F(s)^{-1}$ is holomorphic on $\operatorname{Re}(s) > 1/2$ (see [4, p.300, 15.6 Theorem]).

It follows that $L(s)^3$ has a meromorphic continuation to $\operatorname{Re}(s) > 1/2$ and is holomorphic except for a pole of order 2 at $s = 1$. Moreover,

$$\begin{aligned} \lim_{s \rightarrow 1} L(s)^3 (s-1)^2 &= \lim_{s \rightarrow 1} \frac{L(s)^3}{\zeta(s)^2} (\zeta(s)(s-1))^2 \\ &= \prod_{p:\text{prime}} \frac{(1-p^{-2})^2}{\left(1 - \frac{5}{3}p^{-2} + \frac{10}{27}p^{-3} + \frac{20}{27}p^{-4} - \frac{8}{27}p^{-5} \right)} \\ &= \zeta(2)^{-2} F(1)^{-1}. \end{aligned}$$

By applying Theorem 3.1 to $L(s)$, we have

$$\sum_{n \leq X} \left(\frac{2}{3} \right)^{f(n)} \sim \frac{A}{\Gamma(2/3)} \cdot \frac{X}{\log(X)^{1/3}},$$

where

$$A = \left(\zeta(2)^2 \prod_{p:\text{prime}} \left(1 - \frac{5}{3}p^{-2} + \frac{10}{27}p^{-3} + \frac{20}{27}p^{-4} - \frac{8}{27}p^{-5} \right) \right)^{-\frac{1}{3}}.$$

REFERENCES

- [1] H. Delange. Généralisation du théorème de Ikehara. *Ann. Sci. Ecole Norm. Sup. (3)*, 71:213–242, 1954.
- [2] A.C. Kable. A variation of the Ikehara-Delange Tauberian theorem and an application. *Comment. Math. Univ. St. Pauli*, 57(2):137–146, 2008.
- [3] M. H. Protter. *Basic elements of real analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1998.
- [4] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [5] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.
- [6] N. Wiener. Tauberian theorems. *Ann. of Math. (2)*, 33(1):1–100, 1932.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN

E-mail address: kato.ryo.78w@st.kyoto-u.ac.jp